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On the existence, uniqueness, and analysis of asymptotic stability properties of the solution of retarded differential systems

D. K. Igobi,[*1] M. O. Egwurube[2] W. A. Adesola[3]

ABSTRACT

The special transcendental character of the characteristic equations of the retarded differential system makes it difficult to analyze such systems. Researchers have used various acceptable mathematical techniques to address the issue. In this paper, the convergent properties of an integral equation equivalent of a retarded system were used to establish the existence and uniqueness of the solution of retarded differential equations. A numerical approximating technique was employed in solving the initial value problem of the retarded system and the solution is presented in the form of a finite series. The asymptotic stability properties of this solution were investigated.

INTRODUCTION

In an attempt to analyse real life problems, the concept of mathematical model or formulation of problems are readily employed (Driver, 1977). The mathematical models often chosen are differential equations. Differential equations merely abstract the reality of dynamic systems by disregarding certain physical facts which seem to be of minor influence, such that in complicated physical situations the differential equation does not guarantee the true picture of reality (Kreyszig, 1979). The introduction of functional differential equations (Hale, 1977; Driver, 1977; Cheban, 2002; Asl and Ulsoy, 2000 and Davies, 2006) has helped to address the lapses of the differential equations in modeling dynamic systems.

Retarded equations are special class of functional differential equations with time lag functions incorporated only in the state of the system, which account for the past states as well as the present states (Asl and Ulsoy, 2003). A general retarded functional differential equation is given as,

$$\dot{x}(t) = f\left(t,\ x(t),\ x(t-nh)\right),\quad n = 1,\ 2,\ 3, \ldots, \qquad (1.0)$$

where $x(t)$ is the state of the system at time *t*, $\dot{x}(t)$ is the derivative of the state function with respect to time t, and $x(t-nh)$ is the time lag function, with h > 0 defining the delay interval.

The challenges of analyzing system (1.0) include the establishment of the theory for the existence and uniqueness of the solution, finding an analytic solution, and analyzing the asymptotic stability properties of the solution (Hale and Cruz, 1970). Necessary and sufficient conditions for the existence and uniqueness of the solution of (1.0) are of immense importance, and research to investigate these conditions are found in the works of Hale and Cruz (1970), Hale (1977), Driver (1977), Onwuatu (1993) and Falbo (1995).

Hale (1977) provides necessary and sufficient conditions for global existence and exponential estimates of the solution of non-linear retarded system

$$\left.\begin{aligned} \dot{x}(t) &= L\left(t,\ x(t-h)\right) + b(t), && t > 0 \\ x(t) &= \varphi, && t \in \left[t_0 - h, t_0\right]. \end{aligned}\right\} \qquad (1.1)$$

This is extended to the linear retarded system (1.0) by Driver (1977) and Falbo (1995).

The set back in analyzing system (1.0) lies in the special transcendental character of its characteristic equation which renders the determination of its analytic solution very difficult. Driver (1977), Liu and Mansour (1984) and Lam (1991) employ the concept of exponential estimate in solving the characteristic equation of (1.0), while Hmamed (1986), Han (2001) and Asl and Ulsoy (2003) utilize approximating techniques in achieving their results. Different acceptable techniques have been employed to investigate the necessary and sufficient conditions for asymptotic stability properties of the solution of (1.0) (Hale, 1977; Driver 1977; Liu and Mansour', 1984; Hmamed, 1986; Asl and Ulsoy, 2003; Han, 2001; and Davies ,2006).This paper explores the convergent properties of the integral equation equivalent of (1.0) to establish the existence and uniqueness of solution of the system. Also, a numerical approximating technique is used in solving an initial value problem of the retarded system and solution presented in form of finite series, whose asymptotic stability properties are investigated for each delay interval by utilizing local Lipschitzian condition.

[*]Corresponding author
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[1]Department of Mathematics/Statistics, Cross River State University of Technology, Calabar, Nigeria
[3]Department of Computer Science, Cross River State University of Technology, Calabar, Nigeria
[2]Department of Mathematics/Computer Science, Federal University of Technology, Yola, Nigeria

METHOD OF STUDY

Notations

E^n is an n – dimensional Euclidean space for $n > 0$, with $\|\cdot\|$ as the Euclidean vector norm. $B_H([t_0-h,t], E^n)$ is a Banach space of continuous differentiable function on $[t_0-h, t]$, where $\{h:[t_o-h,t]\to E^n\}$ *and h is continuous* . $\varphi(s)$ is a continuous differentiable function with norm in $B_H([t_o-h,t], E^n)$ defined as $\|\varphi(s)\| = \sup_{t_0-h\le s\le t_0} |\varphi(s)|$, and $x(s) = x(t-h),\ t \ge t_o$ defines the trajectory segment in $B_H([t_o-h,t], E^n)$.

PROBLEM STATEMENT

Consider a general initial value problem of a retarded system of the form

$$\left.\begin{aligned} \dot{x}(t) &= f(t,\ x(t-h)),\quad h>0 \\ x(s) &= \varphi_0,\quad t_o-h\le s\le t, \end{aligned}\right\} \tag{2.0}$$

where $\dot{x}(t)$ is derivative of the state function x(t) with respect to time t, and $x(t-h)$ is a continuous time lag function with $h > 0$ defining the delay interval. For a given initial condition $x(s)=\varphi_0,\ t_0-h\le s\le t$, does system (2.0) admits a unique solution?

Theorem 1

Let $x(t)$ and f be continuous E^n–valued function with domain

$$D = \left\{ \left(x : \|x(s)-\varphi_0(s)\| \ < h,\ s\in[t_0-nh,t]\ n\ge 1 \right) \right\},$$

such that $f:(t_0-nh,t)\to D$ is a contraction in $B_H([t_0-h,t], E^n)$. Then there exists a unique solution of (2.0).

Proof

The integral equation equivalent of system (2.0) is given as

$$\varphi_{n+1}(t) = \varphi_n(t_0) + \int_{t_0-h}^{t} f(s,\ \varphi_n(s))ds. \tag{2.1}$$

Assume $x(t)=\varphi_{n+1}(t)$ is a solution of (2.1) passing through $(\varphi_0(s),t_0),\ (\varphi_{n+1}(s),t) \in B_H([t_0-h,\ t],\ E^n)\ x\ E^n$, then $\dot{\varphi}_{n+1}(t) = f(t,\ \varphi_n(t-h))$. Since f is continuously differentiable on the close interval $[t_0-h,t]$, let there exists a positive real value $m>0$ such that $f(t_1,\ \varphi(t-h))>m$, for any $t_1\in[t_0-h,t]$. Suppose that at $t_1\in[t_0-h,t]$, $f(t_1,\varphi(t_0-h))=\varphi_1(s),\ where\ t_0-h\le s\le t_1$, it implies

$$\varphi_1(s)=f(t_1,\varphi(t_0-h))>m. \tag{2.2}$$

Since $\varphi_1(s)$ is a solution on $[t_0-h,t_1]$, then for every $t_2>t_1$, $t_2\in[t_0-h,t]$, there exists a solution $\varphi_2(s)=f(t_2,\varphi(t_0-h)),\ where\ t_0-h\le s\le t_2$ and $\varphi_2(s)>\varphi_1(s)$. Therefore, for $t_n\in[t_0-h,t]$, $\varphi_n(s)=f(t_n,\varphi(t_0-h))$. Thus we have a sequence of nested solutions $\{\ \varphi_n(s)\ \}$ in the close interval $[t_0-h,t]$. Let f be a contraction in $B_H([t_0-h,\ t],\ E^n)\ x\ E^n$ and for any real constant $0<m_0<1$,

$$\|f(t,\varphi(t-h))\|\le m_0\|\varphi(s)\|,\ t_0-h\le s\le t \tag{2.3}$$

holds. Thus $f(t_1,\ \varphi(t-h)),\ f(t_2,\ \varphi(t-h)),\ .\ .\ .\ f(t_n,\ \varphi(t-h))$ are closer for each $t_1>t_2>\ldots>t_n$.

Assume $\{\ \varphi_n(s)\ \}$ to be a bounded monotone increasing sequence of solution, for any positive value $\varepsilon>0$, there exists at least one positive integer i>0 such that

$$\|\varphi(s)\|-\varepsilon<\varphi_i(s)\le\|\varphi(s)\| \tag{2.4}$$

Now $\varphi_n(s)\le\varphi_{n+1}(s)$ for all n, hence for every n>1, $\varphi_n(s)$ satisfies (2.4), and hence

$$\|\varphi(s)\|-\varepsilon<\varphi_n(s)\le\|\varphi(s)\|+\varepsilon$$

The norm $\|\varphi(s)\| = \sup_{t_0-h\le s\le t}\|\varphi(s)\|$ defines the least upper bound of f , and

$$\lim_{n\to\infty}\varphi_n(s)=\varphi(s),\ t_0-h\le s\le t.$$

Also from (2.1),

$$\lim_{n\to\infty}\varphi_{n+1}(t) = \lim_{n\to\infty}\left(\varphi_n(t_0)+\int_{t_0-h}^{t} f(s,\varphi_n(s))ds\right)$$

$$= \lim_{n\to\infty} \varphi_n(t_0) + \int_{t_0-h}^{t} \lim_{n\to\infty}\left(f(s, \varphi_n(s))ds\right)$$

$$= \lim_{n\to\infty} \varphi_n(t_0) + \int_{t_0-h}^{t} \left(f(s, \lim_{n\to\infty} \varphi_n(s)\right)ds \; = \varphi(t) \quad (2.5)$$

Therefore by (2.4) and (2.5), solution of system (2.0) converge to the $\varphi(t)$, where $\varphi(t)$ is a continuous differentiable function on $B_H\left([t_0 - h, t], E^n\right)$.

Assuming

$\varphi_{m+1}(t) = \varphi(t_0)_m + \int_{t_0-h}^{t} f\left(s, \varphi_m(s)\right)ds$ is another solution of (2.0) on $\left[t_0 - h, t\right]$ such that for any real value k>0 on $\left[t_0 - h, t\right]$

$$\left\|\varphi_{n+1}(t) - \varphi_{m+1}(t)\right\| = \left\|\int_{t_0-h}^{t} f\left(s, \varphi_n(s)\right)ds - \int_{t_0-h}^{t} f\left(s, \varphi_m(s)\right)ds\right\|$$

$$\leq k\left\|V_0\right\| \max_{t_0-h\leq s\leq t}\left\|V(s)\right\| , \quad (2.6)$$

where $\left\|V_0\right\| = \left\|t - (t_0 - h)\right\|$, and $\left\|V(s)\right\| = \left\|\varphi_n(s) - \varphi_m(s)\right\|$.

By the contraction of f in $B_H\left([t_o - h, t], E^n\right) x\ E^N$, k is the Lipschitz constant and f is bounded with a fixed point $\varphi(t)$. Thus (2.5) is the unique solution of (2.0).

NUMERICAL APPROXIMATION PROCEDURE

An analytic solution of (2.0) is not easily obtained unlike its equivalent ordinary differential system. A numerical approximation method is employed on each delay sub-interval, and the solution is presented in form of a finite series. The procedure involve the approximation of solution of the system for each T_i; i = 1, 2, 3, ., ., ., n delay subinterval as the delay h varies on a regular basis. The solution on the preceding interval is use to approximate the solution on the immediate succeeding interval, with the time function (t) depending on the origin of each delay subinterval under consideration

Linearizing the time variants $\left(f(t, x(t-h))\right)$ of (2.0) with respect to *x(t)* results in a simple linear first order retarded equation represented by the initial value problem

$$\left.\begin{aligned} \dot{x}(t) &= a\ x\ (t-h), \quad h > 0 \\ x_0(s) &= \varphi_0, \quad t_0 - h \leq s \leq t, \end{aligned}\right\} \quad (3.0)$$

where ‘a’ is a scalar and $x(s) = \varphi_0$, is the initial value at $t_0 - h \leq s \leq t$. System (3.0) admits a unique solution on $\left[t_0 - h, t\right]$.

Consider (3.0) on T_i sub-interval,

$$T_1: \quad t_0 - h \leq t < t_0$$
$$T_2: \quad t_0 \leq t < t_0 + h$$
$$T_3: \quad t_0 + h \leq t < t_0 + 2h$$
$$T_4: \quad t_0 + 2h \leq t < t_0 + 3h$$
$$\vdots$$
$$T_n: \quad t_0 + (n-1)h \leq t < t_0 + nh,,$$

for *h > 0*, the solution is formulated using the numerical approximation for each delay subinterval as follows.

For

$$T_1: \quad t_0 - h \leq t < t_0$$

$$x_1(t) = a\int x_0(t)dt + c_1,$$
$$= a\int \varphi_0 dt + c_1.$$
$$= a\varphi_0 t + c_1$$

At $t = t_0 - h$, $x_1(t) = \varphi_0$, and substituting for t and $x_1(t)$ in (3.1), we obtained

$$x_1(t) = \varphi_0 + a\varphi_0(t - (t_0 - h)). \quad (3.2)$$

For T_2; $\quad t_0 \leq t < t_0 + h,$

$$x_2(t) = a\int x_1(t)dt + c_2,$$
$$= a\int (\varphi_0 + a\varphi_0(t_0 - (t_0 - h)))dt + c_2.$$

$$a\varphi_0 t + a^2\varphi_0\left(\frac{t^2}{2} - (t_0 - h)t\right) + c_2. \quad (3.3a)$$

At $t = t_0$, $x_2(t) = \varphi_0 + a\varphi_0(t - (t_0 - h))$, and substituting for t and $x_2(t)$ in (3.3a), we obtained,

$$x_2(t) = \varphi_0 + a\varphi_0(t - (t_0 - h)) + a^2\varphi_0\left(\frac{t^2 - t_0^2}{2!} - (t - t_0)(t_0 - h)\right) \quad (3.3b)$$

For T_2; $\quad t_0 + h \leq t < t_0 + 2h,$

$$x_3(t) = a\int x_2(t)dt + c_3,$$

$$= a\int (\varphi_0 + a\varphi_0(t - (t_0 - h)) + a^2\varphi_0\left(\frac{t^2 - t_0^2}{2!} - (t - t_0)(t_0 - h)\right))dt + c_3.$$

At

$t = t_0 - h,\ x_3(t) = \varphi_0 + a\varphi_0(t - (t_0 - h)) + a^2\varphi_0(\frac{t^2 - t_0^2}{2!} - (t - t_0)(t_0 - h)),$

substituting for t and $x_3(t)$ in (3.4a) we obtained

$$x_3(t) = \varphi_0 + a\varphi_0\left((t-(t_0+h)\right) + a^2\varphi_0\left(\frac{t^2-t_0^2}{2!} + (t-(t_0+h)t_0(t_0-h))\right) + a^3\varphi_0(\frac{t^3-(t_0+h)^3}{3!} - (\frac{t^2-(t_0+h)^2}{2!} - (t-(t_0+h)t_0)(t_0-h))). \tag{3.4b}$$

For T_n; $t_0 + (n-1)h \le t < t_0 + nh$,

$$x_n(t) = a\int x_{n-1}(t)dt + c_{n-1}$$
$$= a\int ((\varphi_0 + a\varphi_0((t-(t_0-h)))) + a^2\varphi_0\left(\frac{t^2-t_0^2}{2!}\right) + (t-(t_0-h))t_0(t_0-h))dt +$$
$$+a\int\left(\left(a^3\varphi_0\left(\frac{t^3-(t-h)^3}{3!} - \frac{t^2-(t_0+h)^2}{2!} - (t-(t_0+h)t_0)(t_0-h)\right)\right)\right)dt \ \dots$$
$$+a\int\left(a^{n-1}\varphi_0\left(\frac{t^{n-1}-(t_0+(n-1)h)^{n-1}}{(n-1)!} \dots (t-(t_0-(n-1)h)\dots t_0(t_0-h)\right)\right)dt \quad +c_n$$

At $t = t_0 - (n-1)h,\ \ x_n(t) = x_{n-1}(t),$,

$$x_n = \varphi_0 + a\varphi_0(t-(t_0-h)) + a^2\varphi_0\left(\frac{t^2-t_0^2}{2!} - (t-(t_0+h)t_0(t_0-h))\right)$$
$$+a^3\varphi_0\left(\frac{(t^3-(t_0+h)^3}{3!} + \frac{(t^2-(t_0+2h)^2}{2!} - (t-(t_0+h)t_0(t_0-h)\right) + \dots$$
$$+ a^{n-1}\varphi_0\left(\frac{(t^{n-1}-(t_0+(n-1)h)^{n-1}}{(n-1)!} ----(t-(t_0-(n-1)h)\ \dots t_0(t_0-h)\right)$$
$$+ a^n\varphi_0\left(\frac{t^n-(t_0+nh)^n}{n!} + \frac{t^{n-1}-(t_0+(n-1)h)^{n-1}}{n!} ---(t-(t_0+nh)(t_0-(n-1)h)\dots(t_0-h)\right) \tag{3.5}$$

Stability analysis

Considering the resulting approximate solution of system (3.0) on each T T_i; i = 1, 2, 3, ., ., ., n delay subinterval with a corresponding initial condition as stated below,

$T_1:\quad t_0 - h \le t < t_0$ and $x_0(t_0 - h) = \varphi_0$,

$x_1(t) = \varphi_0 + a\varphi_0(t - (t_0 - h)).$

$T_2: t_0 \le t < t_0 + h$,and

$x_1(t_0) = \varphi_0 + a\varphi_0(t - (t_0 - h))$,

$x_2(t) = \varphi_0 + a\varphi_0(t - (t_0 - h)) + a^2\varphi_0(\frac{t^2 - t_0^2}{2!} - (t - t_0)(t_0 - h))$.

$T_3: t_0 + h \le t < t_0 + 2h,$ and

$x_2(t_0 + h) = \varphi_0 + a\varphi_0(t - (t_0 - h)) + a^2\varphi_0(\frac{t^2 - t_0^2}{2!} - (t - t_0)(t_0 - h))$,

$$x_3(t) = \varphi_0 + a\varphi_0\left((t-(t_0+h)\right) + a^2\varphi_0\left(\frac{t^2-t_0^2}{2!} + (t-(t_0+h)t_0(t_0-h))\right) + a^3\varphi_0(\frac{t^3-(t_0+h)^3}{3!} - (\frac{t^2-(t_0+h)^2}{2!} - (t-(t_0+h)t_0)(t_0-h))).$$

$T_n;\ t_0 + (n-1)h \le t < t_0 + nh$, and

$x_n(t_0 + (n-1)h) = x_{n-1}(t)$,

$$x_n = \varphi_0 + a\varphi_0(t-(t_0-h)) + a^2\varphi_0\left(\frac{t^2-t_0^2}{2!} - (t-(t_0+h)t_0(t_0-h))\right)$$
$$+a^3\varphi_0\left(\frac{(t^3-(t_0+h)^3}{3!} + \frac{(t^2-(t_0+2h)^2}{2!} - (t-(t_0+h)t_0(t_0-h)\right) + \dots$$
$$+ a^{n-1}\varphi_0\left(\frac{(t^{n-1}-(t_0+(n-1)h)^{n-1}}{(n-1)!} ----(t-(t_0-(n-1)h)\ \dots t_0(t_0-h)\right)$$
$$+ a^n\varphi_0\left(\frac{t^n-(t_0+nh)^n}{n!} + \frac{t^{n-1}-(t_0+(n-1)h)^{n-1}}{n!} ---(t-(t_0+nh)(t_0-(n-1)h)\dots(t_0-h)\right)$$

The asymptotic stability properties will now be analyzed for any change in the initial condition $x(s) = \varphi_0,\ t_0 - h \le s \le t$.

Definition Han(2001),

(i) The solution *x(t)* of system (3.0) is Lyapunov stable if for any $\varepsilon > 0$, there exists $\delta = \delta(t - h,\ \varepsilon) > 0$ such that if $\|\varphi(s)\| < \delta$ then $\|x(t - nh, \varphi(s))\| < \varepsilon,\ t_0 - h \le s \le t_0$.

(ii) The solution *x(t)* of (3.0) is asymptotically stable if it is Lyapunov stable, and there exists a $\delta_1 = \delta_1(t - h)$ satisfying $\|\varphi(s)\| < \delta_1$ such that $\|x(t - nh, \varphi(s))\| \to 0$ as $t \to \infty$.

(iii) The solution *x(t)* is uniformly asymptotically stable if it is stable, and furthermore there exists $\delta_2 > 0$ (independent of *t-h*) such that if $\|\varphi(s)\| < \delta_2$, then $\|x(t - nh, \varphi(s))\| \to 0$ as $t \to \infty$.

Theorem 2

Assume $f : B_H \ x \ D \to E^n$ for $D \subset E^n$ is continuous, satisfy local Lipschitzian condition on $[t_0 + nh, t], \ n = 0,1,2...$, and global Lipschitzian condition on $[t_0 - h, \ \infty)$, and is compact in D. The resulted solution $x(t - nh, \ \varphi_0(s))$ of (3.0) is,

(i) Lyapunov stable if for any change in the initial condition $x(t) = \varphi_0(s), \ t_0 - h \le s \le t_0$, the solution $x(t - nh, \ \varphi_0(s))$ remained valid on the entire $[t_0 - h, \ \infty)$.

(ii) Asymptotically stable if limit $\|x(t - nh, \ \varphi_0(s))\| = 0$ for an infinite increment in time (*t*).

Proof

Since(3.0)is continuous on each $[\ t_0 + nh, t]$ for $n = 0, 1, \ 2, \ldots$ and by the approximating technique formulation, let there exist solutions,

$$x(t) = f\left(t - nh, \ \varphi_0(s)\right) \text{and}$$

$$\bar{x}(t) = f\left(t - nh, \ \bar{\varphi}_0(s)\right), \text{ for } t_0 - h \le s \le t_0. \qquad (4.1)$$

satisfying (3.0). If for any pre-determined constant $\varepsilon > 0$ there exists a $\delta = \delta(t - h, \ \varepsilon)$, for $0 \le t - h < \delta < \varepsilon$, such that $\|x(t) - \bar{x}(t)\| < \delta$ holds, it follows that,

$$\| f(t - h, \ \varphi_0(s)) - f(t - h, \ \bar{\varphi}_0(s))\| < \varepsilon. \qquad (4.2)$$

This implies that $x(t - nh, \ \varphi_0(s))$ is valid on $[t_0 - h, \ \infty)$, and is Lyapunov stable.

If δ_1 is a constant, and $0 \le t - h < \delta_1 < \varepsilon$ such that $\delta_1 = \delta_1(t - h)$>0, then (4.1) implies $\|x(t) - \bar{x}(t)\| < \delta_1$. Also since (3.0) is locally Lipschitzian on $[t_0 + nh, t]$, then $x_i(t - nh, \ \varphi_0(s)), \ i = 1, \ 2, \ 3, \ \ldots n$ of (3.2), (3.3) and (3.5) are monotone functional solutions on the bounded interval $[t_0 + (n-1)h, \ t_0 + nh]$.

By Weierstrass – Bolzano concept of boundeness in a close interval,

$$\left\| x_1\left(t_0 + h, \ \varphi_0(s)\right) - \bar{x}_1\left(t_0 + h, \ \bar{\varphi}_0(s)\right)\right\| = \left\| \frac{x\left(t_0 + h, \ \varphi_0(s)\right) - \bar{x}\left(t_0 + h, \ \bar{\varphi}_0)(s)\right)}{2} \right\|,$$

$$\left\| x_2\left(t_0 + 2h, \ \varphi_0(s)\right) - \bar{x}_2\left(t_0 + 2h, \ \bar{\varphi}_0(s)\right)\right\| = \left\| \frac{x\left(t_0 + h, \ \varphi_0(s)\right) - \bar{x}\left(t_0 + h, \ \bar{\varphi}_0(s)\right)}{2^2} \right\|,$$

:

:

$$\left\| x_n\left(t_0 + nh, \ \varphi_0(s)\right) - \bar{x}_n\left(t_0 + nh, \ \bar{\varphi}_0(s)\right)\right\| = \left\| \frac{x\left(t_0 + h, \ \varphi_0(s)\right) - \bar{x}\left(t_0 + h, \ \bar{\varphi}_0(s)\right)}{2^n} \right\|,$$

holds.

Therefore,

$$\left\| x\left(t_0 + h, \varphi_0(s)\right) - \bar{x}\left(t_0 + h, \bar{\varphi}_0(s)\right)\right\| \le \left\| x_1\left(t_0 + h, \varphi_0(s)\right) - \bar{x}_1\left(t_0 + h, \bar{\varphi}_0(s)\right)\right\|$$

$$+ \left\| x_2\left(t_0 + 2h, \varphi_0(s)\right) - \bar{x}_2\left(t_0 + 2h, \bar{\varphi}_0(s)\right)\right\|$$

:

:

$$+ \left\| x_n\left(t_0 + nh, \varphi_0(s)\right) - \bar{x}_n\left(t_0 + nh, \varphi_0(s)\right)\right\|$$

$$= \left\| \frac{x\left(t_0 + h, \ \varphi_0(s)\right) - \bar{x}\left(t_0 + h, \ \bar{\varphi}_0(s)\right)}{2^n} \right\|. \qquad (4.3)$$

By the continuity of f on $[t_0 + nh, t]$, the solutions $x_n(t - h, \ \varphi_0(s))$ and $\bar{x}_n(t - h, \ \varphi \ (s))$ converge to $\varphi(t) \ and \ \bar{\varphi}(t)$ respectively. Thus f forms a compact set in D, and

$$\lim_{\substack{n \to \infty \\ t \to \infty}} \left\| \frac{\varphi(t) - \bar{\varphi}(t)}{2^n} \right\| = 0.$$

(4.4)

Therefore the resulted solution of (3.0) is asymptotically stable.

ILLUSTRATION

Consider the retarded system of the form,

$$\dot{x}(t) = 1 + x(t - 1)$$
$$x_0(1) = 1, \ x_0(t - 1) = 1 \ and \quad t \in [1, \infty) \qquad (5.0)$$

By the numerical approximation method in (3.0),

$x_1(t)$ on $1 \le t < 2$,

$$x_1(t) = \int \ (1 + x_0(t - 1))dt \ + c_1$$
$$= \int 2dt \ + c_1 \qquad (5.1)$$

Solving (5.1) at an initial state of $x_0(1) = 1$,

$x_1(t) = 2t - 1 \qquad = 1 - 2[-(t-1)]$, (5.2)a

and $x_1(t-1) = 1 - 2[-(t-2)]$. (5.2b)

Considering $x_2(t)$ on $2 \le t < 3$,

$$x_2(t) = \int (1 + x_1(t-1))dt + c_2$$

$$= \int (1 + 2(t-2) + 1)dt + c_2 \qquad = t^2 - 2t + c_2 \quad (5.3)$$

Solving (5.3) at an initial state of $x_1(2) = 1 - 2[-(t-1)]$,

$$x_2(t) = t^2 - 2t + 3$$

$$= 1 - 2\left[-(t-1) - \frac{(t-2)^2}{2!}\right], \qquad (5.4a)$$

And

$$x_2(t-1) = 1 - 2\left[-(t-2) - \frac{(t-3)^2}{2!}\right]. \qquad (5.4b)$$

Considering $x_3(t)$ on $3 \le t < 4$,

$$x_3(t) = \int (1 + x_2(t-1))dt + c_3$$

$$= \int (1 + 2\left[(t-2) + \frac{(t-3)^2}{2!}\right] + 1)dt + c_3$$

$$= \int (t^2 - 4t + 7)dt + c_3. \qquad (5.5)$$

Solving (5.5) at an initial state of

$$x_2(3) = 1 - 2\left[-(t-1) - \frac{(t-2)^2}{2!}\right],$$

$$x_3(t) = \frac{t^3}{3} - 2t^2 + 7t - 9$$

$$= 1 - 2\left[-(t-1) - \frac{(t-2)^2}{2!} - \frac{(t-3)^3}{3!}\right], \qquad (5.6a)$$

and

$$x_3(t-1) = 1 - 2\left[-(t-2) - \frac{(t-3)^2}{2!} - \frac{(t-4)^3}{3!}\right]. \qquad (5.6b)$$

Therefore for $x_n(t)$ on $n \le t < n+1$

$$x_n(t) = \int (1 + x_{n-1}(t-1))dt + c_n$$

$$x_n(t) = 1 - 2\left[-(t-1) - \frac{(t-2)^2}{2!} - \frac{(t-3)^3}{3!} - \frac{(t-4)^4}{4!} - \ldots - \frac{(t-n)^n}{n!}\right],$$

and

$$x_n(t-1) = 1 - 2\left[-(t-2) - \frac{(t-3)^2}{2!} - \frac{(t-4)^3}{3!} - \frac{(t-5)^4}{4!} - \ldots - \frac{(t(n+1))^n}{n!}\right].$$

Indeed, the general solution of (5.0) is expressed as

$$x_n(t) = 1 - 2\exp(-(t-i)), \qquad 1 \le i \le n, \qquad (5.7a)$$

where *i* measures the change in time lag (h) and

$$x_n(t-1) = 1 - 2\exp(-(t-(i+1))). \qquad (5.7b)$$

The result (5.7) above is comparative to the iterative method for an equivalent ordinary differential equation of (5.0)

Also by theorem 2 , the solution *x(t)* is asymptotically stable, if $\|x_1(t) - x_0(t)\| < \delta$ such that $\lim_{t\to\infty} \|x_1(t) - x_0(t)\| = 0$

Using solutions (5.2a), (5.4a) and (5.6a) with the initial conditions, then

$$\lim_{t\to\infty} \|x_{n+1}(t) - x_n(t)\| = \lim_{t\to\infty} 2\exp(-(t-i)) = 0$$

This implies that every solution of system 5.0 is asymptotically stable.

CONCLUSION

The theorem on the existence and uniqueness of solution of delay retarded system is established and proved using the convergent properties of an integral equivalent of the retarded system. A numerical approximation method is employed to find solution of the system equation for each n-subinterval, and the asymptotic stability property is analyzed. Results obtained from illustration proved the suitability of this analysis.

REFERENCES

Asl, F. M. and Ulsoy A. G. (2003). Analysis of system of linear delay differential equations, *Journal of Dynamic Systems, Measurement and Control*, (125):215-223.

Cheban D. N. (2002). Uniform exponential stability of linear almost periodic systems in banach spaces, *Electronic Journal of Differential Equation,* (20):29.

Davies, I. (2006). Criteria for stability and boundedness of linear delay system, *Journal of the Mathematical Association of Nigeria,* 33(2B): 419-426.

Driver, R. D. (1977). Ordinary and *delay differential equations*, Sprmger-Verlag, New York.

Falbo, C. E. (1995). Analytic and numerical solutions to delay differential equations, *Joint Meeting of Northern and Southern California Sections of the Mathematical Association of America.*

Hale, J. K. (1977). *Theory of differential equation*, Sprnger-Verlag, New-York.

Hale, J. K. and Cruz, M. A. (1970). Existence, uniqueness and continuous dependence for hereditary systems. *Annali di Mathematica Pura ed Applicata* 85:63-82.

Han, Q. L. (2001). On delay-dependent stability for neutral delay. *Differential Systems (II)* 4:965-976.

Hmamed, A. (1986). Stability conditions of delay-differential system, *Int. J. Control*, 43(2):455-463.

Kreyszig, E. (1979). *Advanced engineering mathematics*, John Wiley and Sons, New York.

Lam, J. (1991). Balanced realization of pade approximation of e^{-ST} (All-pass Case). *IEEE Trans. Autom Control* 39: 1096-1110.

Liu, X. Y. and Mansour, M. (1984). Stability test and stability conditions for delay differential systems. *Int. J. Control* 39(6): 1229-1242.

Onwuatu, J. U. (1993). Null controllability of non-linear infinite neutral-system. *KYBERNE TIKA* 29:484-487.